Computational complexity of optimizing defect braiding quantum circuits by reordering qubits

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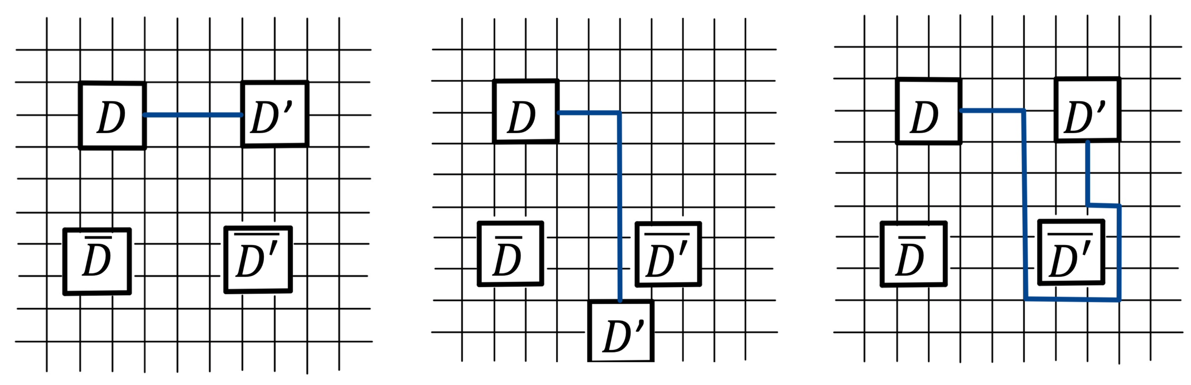
We have formulated a part of the optimization problem of defect braiding quantum circuits for fault-tolerant quantum computation on surface codes and showed that the computational complexity is NP-hard.

Background

Surface codes [1] have excellent threshold performance and are highly feasible since they have local and small stabilizer generators. They are expected to be utilized in fault-tolerant quantum computation (FTQC). However, the realization of FTQC requires many resources. To realize universal FTQC, continuous unitary gates will be decomposed and approximated by a finite set of quantum gates called a universal gate set [2], which significantly increases the depth of the circuit. Defect braiding [3] and lattice surgery [4] have been proposed as methods to realize universal gate sets on surface codes, but using these methods increases the execution time of each gate compared to the uncoded case. To reduce these overheads, optimization of FTQC quantum circuits is crucial. Several practical optimizations of defect braiding quantum circuits have been proposed [5,6]. It is helpful to formulate the optimization problem and show the computational complexity to achieve further optimization. This allows us to estimate the optimization problem's cost and clarify the compiler's requirements.

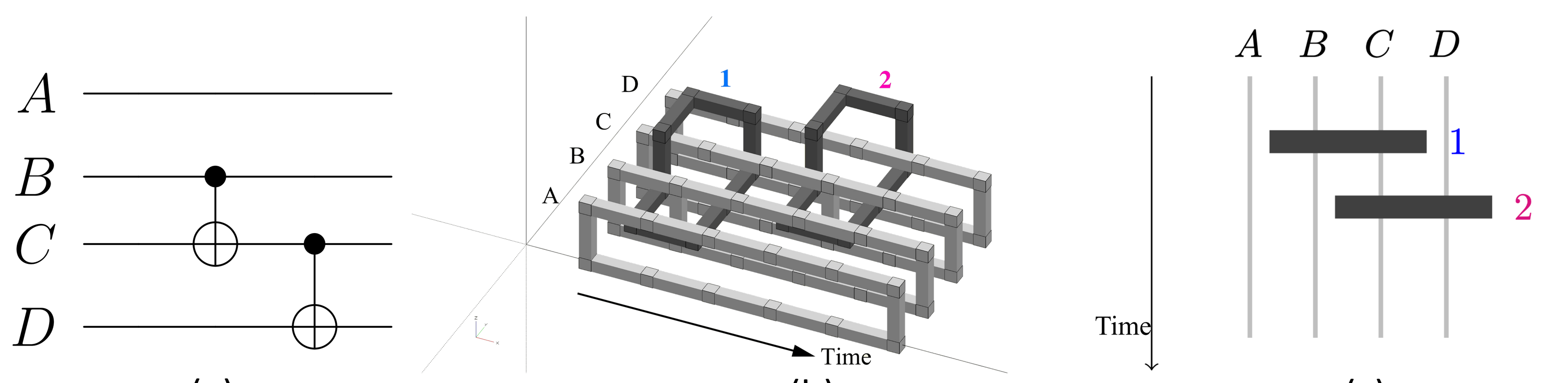
Defect Braiding

Multi-qubit gates are necessary to realize universal gate sets in FTQC. Defect braiding is known as a method to realize CX gates on surface codes. First, a pair of defects is prepared on the surface code as shown in the figure, which corresponds to a logical qubit. Defect braiding is an operation to realize logical CX gates by moving them so that they wrap around each other.



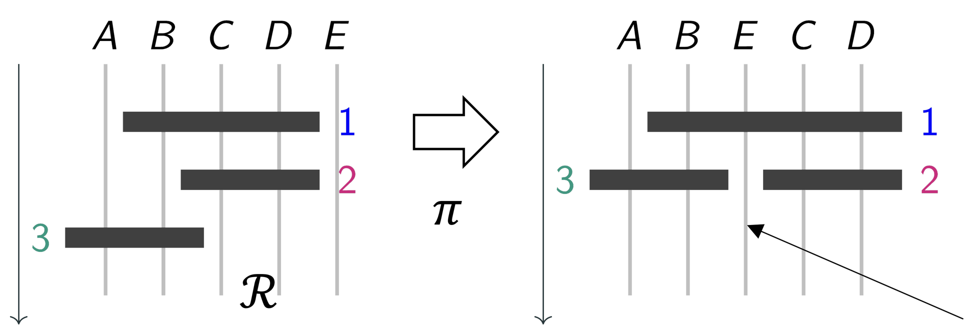
Defect braiding operation realizes a logical gate $CX(D,\overline{D})$ between two defect pair qubits $D,\overline{D}$.

For example, a simple quantum circuit is shown in (a). A defect braiding quantum circuit to realize (a) is given by (b). This study deals with the optimization of a circuit like (c) (called a 1D defect braiding quantum circuit) in which the 3D defect braiding circuit is mapped onto a 2D plane, and the qubits are arranged in one dimension, ignoring the direction of the CX gates for the sake of simplicity.



(a) (b) (c)

Example of the optimization:
The height is reduced from 3 to 2 by rearranging the qubits.



enough margin between gate 2&3
"X_2 and X_3 can be arranged into one row"

Gates $\mathcal{R}$: $1=\{B,C,D\}$ $2=\{C,D\}$ $3=\{A,B\}$
Partial order: $2 \succ 1, 3 \succ 1$
$\pi: \{A,B,C,D,E\} \to \{A,B,E,C,D\}$

Reference

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Problem Definition

Let $[n]=\{1,2,\dots,n\}$ denote the set of n logical qubits. We are given a family $\mathcal{R}$ of gates, where each gate $X \in \mathcal{R}$ is a subset of $[n]$. We are also given a partial order $\succeq$ on $\mathcal{R}$, which represents the order of operators on the same logical qubit.

Conditions

For a permutation $\pi: [n] \to [n]$ of logical qubits, we say that gates $X_1,\dots,X_l \in \mathcal{R}$ can be arranged into one row with respect to π if, for any distinct $i,j \in \{1,2,\dots,l\}$, either

$$\max\{\pi(x) \mid x \in X_i\} + 1 < \min\{\pi(x') \mid x' \in X_j\}$$

or

$$\max\{\pi(x') \mid x' \in X_j\} + 1 < \min\{\pi(x) \mid x \in X_i\}$$

holds. This condition means that any pair of gates in the same row have no overlap, and furthermore there is a margin between them.

We say that the family $\mathcal{R}$ of gates can be packed with height $h \in \mathbb{Z}_+$ according to a partial order $\succeq$ if there exists a pair (π,μ) of a permutation $\pi: [n] \to [n]$ and a function $\mu: \mathcal{R} \to [h]$ satisfying the following two conditions:
1. (the horizontal condition) For each $i \in [h]$, the gates in $\{X \mid \mu(X) = i\}$ can be arranged into one row with respect to a permutation $\pi: [n] \to [n]$, and
2. (the vertical condition) For any two distinct gates $X, X' \in \mathcal{R}$, if $X \cap X' \neq \emptyset$ and $X \succ X'$, then $\mu(X) > \mu(X')$.

For a packing (π,μ), we call $\mu(X)$ the level of $X \in R$.
Here we define our problem.

Min-Braiding Problem
Given a set $\mathcal{R}$ of gates, a partial order $\succeq$ on $\mathcal{R}$, and a positive integer h, output Yes if $\mathcal{R}$ can be packed according to $\succeq$ with height at most h.

Hardness Result

Theorem **Min-Braiding** is an NP-Complete problem.

Summary of the proof
- Assume there exists an algorithm $\mathcal{A}$ which can solve **Min-Braiding** efficiently.
- For any instance of a known hard problem $\mathcal{P}$, Min-Braiding is at least more difficult than $\mathcal{P}$ if it can be solved using $\mathcal{A}$.

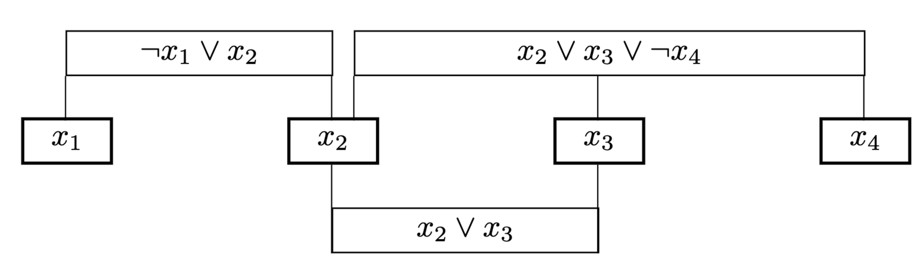
One widely known difficult (NP-complete) problem is the 3SAT.

Definition (3SAT) Given 3-CNF boolean formula ϕ, output *Yes* if there exists an interpretation that satisfies ϕ.

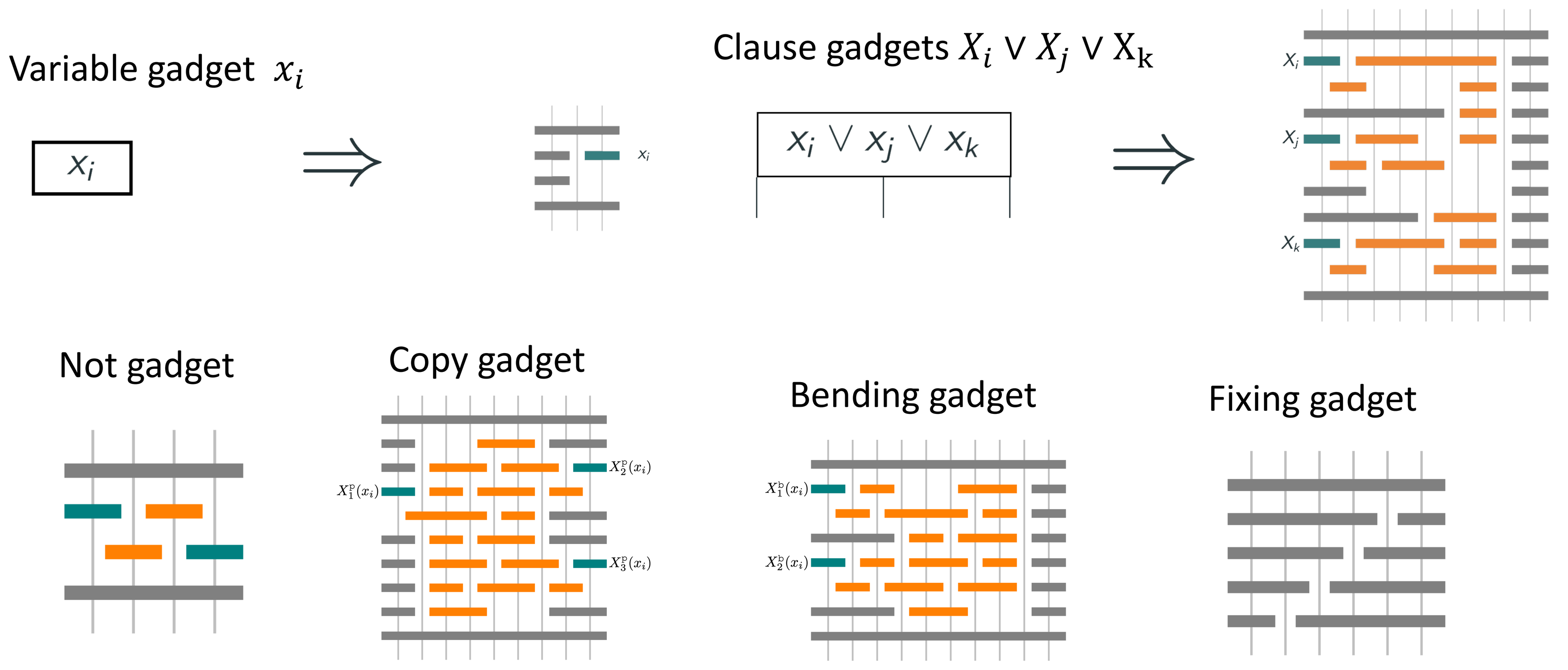
Example:
$\phi = (\neg x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$
clause: $c_1 = (\neg x_1 \vee x_2), c_2 = (x_2 \vee x_3), c_3 = (x_2 \vee x_3 \vee \neg x_4)$
interpretation: $(x_1,x_2,x_3,x_4) = (0,0,1,0)$

We use the polynomial reduction from a NP-Complete problem called **PlanarRectLinear 3SAT** [7,8], which is a special case of 3SAT that can be drawn as a graph $G(\phi)$ satisfying the following:
1. Vertices in $G(\phi)$ corresponding to variables x and clause c are drawn by rectangles whose sides are parallel to axis
2. Rectangles corresponding to variables are drawn on a horizontal line
3. Vertical line segments represent attributions to clause and, and edges do not cross each other.

Example: $G(\phi)$ of $\phi = (\neg x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4)$



Using six different transformations, **PlanarRectLinear 3SAT** can be transformed into **Min-Braiding** in polynomial time.



Since **Min-Braiding** is NP-complete, the problem of finding the optimal packing (π,μ) is NP-hard. Therefore, it is required to design heuristic algorithms or find less generalized optimization methods. This is consistent with known results for Lattice Surgery [9].

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